- For each question you must show your working whenever it is required in order to get full marks.
- When working is required answers only will not be awarded full marks.
- **There is one exception.** If you need to row reduce a matrix you are only required to give the reduced matrix with no working. You may use any application you like to obtain it.

1. (6 marks)

(a) Suppose that

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} -4 \\ -6 \\ -10 \end{pmatrix}, \text{ and } \mathbf{v_3} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

Either show that the collection of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent, or show that it is linearly independent.

(b) Suppose that

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ -3 \\ -3 \\ 0 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \end{pmatrix}.$$

Either show that the collection of vectors $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is linearly dependent, or show that it is linearly independent.

2. (6 marks)

Let

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -2 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} -2 \\ -6 \\ -6 \\ 2 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} -1 \\ -2 \\ -4 \\ 0 \end{pmatrix}, \text{ and } \mathbf{v_5} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

and let

$$A = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}\ \mathbf{v_5}] = \begin{pmatrix} 1 & 1 & -2 & -1 & 1 \\ 4 & 2 & -6 & -2 & 1 \\ 2 & 4 & -6 & -4 & -1 \\ -2 & 0 & 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for col(A), the column space of matrix A.

(b) Express the vector $\mathbf{v_3}$ as a linear combination of the basis vectors found in (a).

3. (7 marks)

Suppose

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 & -4 & 2 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -4 \end{pmatrix}.$$

(a) Write down a basis for col(A). What is the rank of A?

(b) What is the set of all possible vectors $\mathbf{b}$ such that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

4. (12 marks)

(a) Let

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 0 & -6 & 2 \\ 1 & -10 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{-4}{3} \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Find a basis for null(A).

(ii) Consider the linear system

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & -6 & 2 \\ 1 & -10 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Find one solution to this system.

(iii) Find the general solution to the linear system in (ii), expressed as a vector plus a vector space.

(b) Consider the linear system

$$\begin{pmatrix} 1 & 2 & -2 & -1 & 0 \\ 2 & 3 & -5 & 1 & -7 \\ 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & 5 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \end{pmatrix}.$$

Given that

$$\left(\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 0 & 3 \\ 2 & 3 & -5 & 1 & -7 & 4 \\ 0 & 1 & 1 & -1 & 1 & 2 \\ -1 & 1 & 5 & -1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -4 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

find the general solution to this linear system. Express your answer in the form of a vector plus a vector space.

5. (4 marks)

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) State the dimension of the column space of A.

(b) State rank(A).

(c) State the nullity of A.

(d) Which of the sets: col(A), null(A) and the general solution to $A\mathbf{x} = \mathbf{b}$, are vector spaces?

6. (12 marks)

(a) Is the set of vectors:

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix} \right\}$$

orthogonal? Give a reason for your answer.

(b) Suppose S is the vector space spanned by the set of vectors $\{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$\mathbf{u_1} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Use the Gram-Schmidt process to obtain an orthonormal set of vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ that spans S.

(ii) Can the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ be expressed as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$?

7. (2 marks)

Challenge Question: Consider the set V of all polynomials with real coefficients - this set is a vector space over $\mathbb{R}$ if operations of addition and scalar multiplication are defined in the "standard" way. (You don't have to prove that V is a vector space - take it as a given). Consider a subset W of V consisting of all polynomials of degree less or equal to 2. Is W a subspace of V? Can you give an example of a basis for W? What is the dimension of W? State clear reasons for all your answers.

8. (1 marks)

Write a paragraph stating which question from this assignment you found most challenging and why.